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ON BOUNDS AND APPROXIMATE SOLUTIONS FOR A CLASS OF TRANSIENT CREEP PROBLEMS†

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Abstract—The problem of primary and secondary creep in a circular cylinder undergoing torsion is reduced to a nonlinear integral equation of the same form as one previously obtained in [1] for pure bending in beams. *A priori* torsion bounds are then immediate from the results of [1]. A generalized secondary creep equation is then proposed which contains as special cases the above-mentioned pure bending and torsion equations as well as those for cylindrical and spherical pressure vessels. Monotone sequences of approximate solutions for this equation are derived under the assumption of a very general secondary creep law.

1. INTRODUCTION

In [2], a nonlinear integral equation was derived which governs primary creep in cylindrical and spherical pressure vessels subject to a nondecreasing internal pressure. On the basis of this equation, *a priori* upper and lower bounds were obtained for various quantities of physical interest. The strain-hardening law of Odqvist and Hult[3] was assumed (see (1.6) below).

In [1], the Saint-Venant theory of pure bending was extended to primary and secondary creep, resulting in an equation very similar to that of [2]. Applying to this equation methods which constitute a refinement of those of [2], the first author was again able to obtain bounds for primary creep subject to the Odqvist-Hult law, and for generalized secondary creep.

This class of creep bounds hold for all time but would appear to be of computational interest mainly for time periods which are short compared with the time necessary to reach the steady state. There are, however, considerations under which such periods are important involving such things as damage prediction, creep buckling and the design of components which are intended to have short creep lifetimes.

In Section 2 of this paper, the Saint-Venant torsion problem for hollow or solid circular cylindrical bars subject to a positive nondecreasing torque is reduced to an integral equation (2.18) which is mathematically equivalent to the basic equation (2.24) of [1]. This enables us to apply to the torsion problem results of [1] involving not only bounds, but also limiting stress states as $t\to\infty$ and qualitative information on the shape of the shear profile, such as monotonicity (2.20) (a) and convexity (2.20) (b). Such information would appear to be useful in evaluating the quality of computer output and in detecting spurious nonlinear effects.

Section 3 is confined to the study of secondary creep and deals with the construction of approximate solutions for (3.1). The latter equation covers, as special cases, the secondary creep behavior of pressure vessels, beams subject to pure bending, circular cylinders under torsion, and probably various other problems as well. Two monotone, uniformly convergent sequences of functions $\{s_k\}$ and $\{\sigma_k\}$ are defined recursively, each of which converges to the unique continuous solution s of (3.1). The monotone nonincreasing sequence $\{s_k\}$ furnishes an infinite collection of upper bounds the first two of which, s_0 and s_1, are easily computed.

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Similarly, $\{\sigma_k\}$ comprises a monotone nondecreasing family of lower bounds of which σ_0 and σ_1 are easily computed. The two approximation schemes yield not only bounds, but also an existence theorem for (3.1) and a numerical method for its integration.

These results generalize the work of [4] on upper bound and [5] on lower bound approximate solutions for secondary creep in pressure vessels. However, although [4 and 5] used a power law for creep, in the present paper, we assume only that the creep constitutive function satisfy (3.6). Thus, even for the pressure vessel case a substantial generalization is achieved. It is still necessary to require that the input function $N(t)$, interpretable as a pressure in some cases, as a moment in others, be positive and nondecreasing with time.

As in [1 and 2], the infinitesimal strains ϵ_{ij} are assumed to have the form

$$\epsilon_{ij} = \epsilon_{ij}^{(e)} + \epsilon_{ij}^{(c)}, \tag{1.1}$$

where $\epsilon_{ij}^{(e)}$ and $\epsilon_{ij}^{(c)}$ denote elastic strains and creep strains respectively. These are related to the stresses σ_{ij} by the equations†

$$\epsilon_{ij}^{(e)} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}], \tag{1.2}$$

$$\epsilon_{ij}^{(c)}\big|_{t=0} = 0 \tag{1.3a}$$

$$\dot{\epsilon}_{ij}^{(c)} = \frac{F(\sigma_e)}{[\epsilon_e^{(c)}]^m}\cdot s_{ij}, \qquad t>0. \tag{1.3b}$$

Here s_{ij} stands for the deviatoric components of the stress, σ_e is the effective stress and $\epsilon_e^{(c)}$ is the effective creep strain. They are defined by the formulas

$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3}\sigma_{kk}, \tag{1.4}$$

$$\sigma_e = \sqrt{\left(\frac{3}{2}s_{ij}s_{ij}\right)}, \qquad \epsilon_e^{(c)} = \sqrt{\left(\frac{2}{3}\epsilon_{ij}^{(c)}\epsilon_{ij}^{(c)}\right)}. \tag{1.5}$$

For $m=0$, (1.3b) gives a generalized secondary creep law; for $m>0$, (1.3b) generalizes the primary creep law

$$\dot{\epsilon}_{ij}^{(c)} = \frac{3}{2}\frac{K\sigma_e^{n-1}}{[\epsilon_e^{(c)}]^m}\cdot s_{ij} \tag{1.6}$$

of Odqvist and Hult[3].

It is assumed that

$$E>0, \quad -1<\nu\le\frac{1}{2}, \quad m\ge 0, \tag{1.7}$$

that F is C^2, and that for $z>0$

$$F(z)>0, \qquad \frac{d}{dz}[zF(z)]>0, \tag{1.8}$$

$$zF'(z)\ge mF(z). \tag{1.9}$$

Notice that, in the special case of the creep law (1.6), where

$$F(z) = \frac{3K}{2}z^{n-1} \quad (K>0), \tag{1.10}$$

†Subscripts have the range 1, 2, 3, δ_{ij} stands for the Kronecker delta, and summation over repeated indices is implied. We shall also use a superposed dot to denote differentiation with respect to time. Points in three-space are designated either (x_1, x_2, x_3) or (x, y, z).

(1.9) reduces to

$$n \geq m + 1. \tag{1.11}$$

2. BOUNDS FOR PRIMARY CREEP

We consider a hollow or solid cylinder of length l whose cross section has inner radius $r = a \geq 0$ and outer radius $r = b$. Cylindrical coordinates will be used with the z-axis taken along the center line of the cylinder. The lateral boundary conditions are

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0 \quad (r = a \text{ or } b,\ t > 0), \tag{2.1}$$

and the relaxed end conditions on $z = l$ become

$$\int_0^{2\pi}\int_a^b (\cos\theta\sigma_{zr} - \sin\theta\sigma_{z\theta}) r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_a^b (\cos\theta\sigma_{z\theta} + \sin\theta\sigma_{zr}) r\,\mathrm{d}r\,\mathrm{d}\theta = 0, \tag{2.2a}$$

$$\int_0^{2\pi}\int_a^b \sigma_{zz} r\,\mathrm{d}r\,\mathrm{d}\theta = 0, \tag{2.2b}$$

$$\int_0^{2\pi}\int_a^b \sigma_{zz}\cos\theta r^2\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_a^b \sigma_{zz}\sin\theta r^2\,\mathrm{d}r\,\mathrm{d}\theta = 0, \tag{2.2c}$$

$$\int_0^{2\pi}\int_a^b \sigma_{z\theta} r^2\,\mathrm{d}r\,\mathrm{d}\theta = M(t). \tag{2.2d}$$

It is assumed that $\dot{M}$ is continuously differentiable on $[0, \infty)$ and that

$$M > 0, \quad \dot{M} \geq 0 \quad (0 < t < \infty). \tag{2.3}$$

In cylindrical coordinates, the basic kinematic assumption of the Saint-Venant torsion theory for beams with circular cross sections takes the form

$$u_r = u_z = 0, \qquad u_\theta = \alpha(t) rz. \tag{2.4}$$

Here $\alpha(t)$ is the time-dependent specific angle of twist. Substituting (2.4) into the infinitesimal strain-displacement relations (see, e.g. [6], p. 183), we obtain

$$\begin{gathered} \epsilon_{rr} = \epsilon_{\theta\theta} = \epsilon_{zz} = \epsilon_{r\theta} = \epsilon_{rz} = 0, \\ \epsilon_{\theta z} = \frac{\alpha r}{2}. \end{gathered} \tag{2.5}$$

By inspection of the strain–stress relations (1.1), (1.2), (1.3), we see that it is consistent with the above strain field to assume that

$$\sigma_{rr} = \sigma_{\theta\theta} = \theta_{zz} = \theta_{r\theta} = \sigma_{rz} = 0. \tag{2.6}$$

With this assumption, conditions (2.1), (2.2b) and (2.2c) are immediately satisfied. Moreover, in the absence of body forces, the quasistatic equations of motion ([6], p. 184) reduce to

$$\frac{\partial}{\partial z}\sigma_{\theta z} = \frac{\partial}{\partial\theta}\sigma_{\theta z} = 0$$

so that we may set

$$\sigma_{\theta z} \equiv \sigma(r, t). \tag{2.7}$$

This fact, together with (2.6), implies that (2.2a) is met. Also, (2.2d) becomes

$$2\pi \int_a^b \sigma(r,t) r^2 \, dr = M(t). \tag{2.2dr}$$

The only strain–stress relations not trivially satisfied are

$$\epsilon_{\theta z} = \frac{(1+\nu)}{E}\sigma + \epsilon_{\theta z}^{(c)} \tag{2.8}$$

$$\epsilon_{\theta z}^{(c)}|_{t=0} = 0, \tag{2.9}$$

$$\dot{\epsilon}_{\theta z}^{(c)} = \frac{F(\sigma_e)}{[\epsilon_e^{(c)}]^m}\sigma. \tag{2.10}$$

By (1.4), (1.5), (2.5), (2.6) and (2.7),

$$\sigma_e = \left(\frac{3}{2}\right)^{1/2}|\sigma|, \qquad \epsilon_e^{(c)} = \left(\frac{2}{3}\right)^{1/2}|\epsilon_{\theta z}^{(c)}|. \tag{2.11}$$

Because of the assumptions (2.3), it is physically plausible to require that

$$\sigma > 0, \qquad \epsilon_{\theta z}^{(c)} > 0 \quad (t > 0). \tag{2.12}$$

It then follows from (2.10) and (2.11) that

$$\epsilon_{\theta z}^{(c)} = \left[\int_0^t G(\sigma)\, d\tau\right]^{1/(m+1)}, \tag{2.13}$$

where

$$G(\sigma) \equiv (m+1)\left(\frac{3}{2}\right)^{m/2} F\left[\left(\frac{3}{2}\right)^{1/2}\sigma\right]\sigma. \tag{2.14}$$

Equations (2.5), (2.8) and (2.13) now yield

$$\frac{\alpha(t) r}{2} = \frac{(1+\nu)}{E}\sigma + \left[\int_0^t G(\sigma)\, d\tau\right]^{1/(m+1)}. \tag{2.15}$$

Multiplying both sides of (2.15) by r^2, integrating from a to b, and applying (2.2dr), we obtain for the specific angle of twist the representation

$$\alpha(t) = \frac{2}{I}\left(\frac{(1+\nu)}{2\pi E}M(t) + \int_a^b \left[\int_0^t G(\sigma)\, d\tau\right]^{1/(m+1)} \xi^2 \, d\xi\right) \tag{2.16}$$

where

$$I = \int_a^b \xi^3 \, d\xi. \tag{2.17}$$

Substitution of this expression into (2.15) leads to the basic creep equation

$$\frac{(1+\nu)}{E}\sigma = \frac{r}{I}\left(\frac{(1+\nu)}{2\pi E}M(t) + \int_a^b \left[\int_0^t G(\sigma)\, d\tau\right]^{1/(m+1)} \xi^2 \, d\xi\right) - \left[\int_0^t G(\sigma)\, d\tau\right]^{1/(m+1)}. \tag{2.18}$$

This is mathematically the same equation as (2.24) of [1]. It suffices to take $k = \xi^2$ in (2.24) of [1] and to identify $M/2\pi$ of (2.18) above with $-M$ of (2.24) [1] and $E/(1+\nu)$ with E. In [1], the sign convention was such that $-M$ was positive and nondecreasing. A further correspondence enters when adapting pure bending results that are expressed in terms of a particular creep law.

For instance, if the power law (1.10) is used then

$$G(\sigma) = (m+1)\left(\frac{3}{2}\right)^{(m+n+1)/2} K\sigma^n$$

in the torsion problem, and

$$G(\sigma) = (m+1)K\sigma^n$$

for pure bending.

As a first application of these correspondences, let us assume the power law (1.10). Then, by (2.29) of [1], we obtain for the limiting shear field as $t \to \infty$,

$$\sigma(r, \infty) = \frac{M(\infty) r^{(m+1)/n}}{2\pi \int_a^b \xi^{[(m+1)/n]+2}\, d\xi}. \tag{2.19}$$

Formulas for $\sigma(r, \infty)$ with the assumption of other creep laws may be derived by the procedures indicated in [1].

From (3.5), (3.6) and (3.20) of [1] with associated constitutive assumptions, we obtain the inequalities

$$\text{(a)}\ \frac{\partial \sigma}{\partial r} \geq 0, \quad \text{(b)}\ \frac{\partial^2 \sigma}{\partial r^2} \leq 0 \quad (a < r \leq b,\ t \geq 0) \tag{2.20}$$

$$\text{(a)}\ \frac{\partial}{\partial r}\left(\frac{\sigma}{r}\right) \leq 0, \quad \text{(b)}\ \frac{\partial}{\partial r}\left[\left(\frac{\phi}{r}\right)^{1/(m+1)}\right] \geq 0 \quad (a < r \leq b,\ t \geq 0), \tag{2.21}$$

where

$$\phi(r, t) = \int_0^t G(\sigma(r, \tau))\, d\tau. \tag{2.22}$$

As in [1], (2.20) (a) uses only (1.8), while (2.20) (b) is readily verified only with the added restrictions

$$m = 0, \quad G'' \geq 0. \tag{2.23}$$

As for (2.21), (a) and (b), which are equivalent, the proof in [1] holds when, in addition to (1.8) we assume either $m = 0$ or the Odqvist–Hult power law (1.6) subject to (1.11).

From (2.20) (a) and (2.21) (b) upper and lower bounds for σ can be obtained by the same method as that used for upper bounds in [1]. The only difference arises when $a > 0$, in which case a positive lower bound exists. It is found that

$$\frac{aM(t)}{2\pi I} \leq \sigma(r, t) \leq \frac{bM(t)}{2\pi I}. \tag{2.24}$$

In the special case where M is constant, (2.24) simply says that the creep shear stress never goes outside the upper and lower bounds of the initial elastic shear stress. The bounds (2.24) can be used as in [1] to derive short-time upper and lower bounds which tend to the exact solution as $t \to 0$. It follows that (e.g. taking $a \neq 0$)

$$\sigma \leq \frac{Mr}{2\pi I} + \frac{Er}{1+\nu}\left\{b^{-1}\left[\int_0^t G\left(\frac{bM}{2\pi I}\right) d\tau\right]^{1/(m+1)} - a^{-1}\left[\int_0^t G\left(\frac{aM}{2\pi I}\right) d\tau\right]^{1/(m+1)}\right\}, \tag{2.25}$$

$$\sigma \geq \frac{Mr}{2\pi I} - \frac{Er}{1+\nu}\left\{b^{-1}\left[\int_0^t G\left(\frac{bM}{2\pi I}\right) d\tau\right]^{1/(m+1)} - a^{-1}\left[\int_0^t G\left(\frac{aM}{2\pi I}\right) d\tau\right]^{1/(m+1)}\right\}. \tag{2.26}$$

3. SUCCESSIVE APPROXIMATIONS FOR SECONDARY CREEP

Consider the equation

$$s(r,t)=\frac{r^l}{I}\left(N(t)+\int_0^t\int_a^b H(s)q(\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau\right)-\int_0^t H(s)\,\mathrm{d}\tau\ (a\le r\le b,\ t\ge 0) \tag{3.1}$$

where

$$I=\int_a^b \xi^l q(\xi)\,\mathrm{d}\xi. \tag{3.2}$$

Due to the observations of the previous section, it is clear that (3.1) contains as special cases the secondary creep ($m=0$) equations for both torsion and pure bending. In the case of torsion, we have, by comparison with (2.18),

$$s=\frac{(1+\nu)\sigma}{E},\quad l=1,\quad q(\xi)=\xi^2,\quad N=\frac{(1+\nu)M}{2\pi E},$$
$$H(s)=G\left(\frac{Es}{1+\nu}\right). \tag{3.3}$$

For pure bending we use (2.24) of [1] to get

$$s=\frac{\sigma}{E},\quad l=1,\quad N=-\frac{M}{EI},\quad H(s)=G(Es),\quad q(\xi)=k(\xi),\ a=0,\ b=c. \tag{3.4}$$

The introduction of the parameter l enables us to include other geometries as well, e.g. cylindrical ($l=-2$) and spherical ($l=-3$) pressure vessels (see eqn (2.36) of [7]).

It is assumed that N is C^1 on $[0,\infty)$, q is continuous on $[a,b]$, and that

$$N>0,\ \dot N(t)\ge 0\ (0<t<\infty),\ q(r)>0\ (0<r<a),\ l\neq 0,\ 0<I<\infty. \tag{3.5}$$

The basic constitutive assumption for this section is that H be C^2 in $[0,\infty)$, and that

$$H(0)=0,\ H'(z)>0,\ H''(z)>0\ (0<z<\infty). \tag{3.6}$$

It follows from (3.6) that

$$H'(z)z-H(z)>0\ (0<z<\infty). \tag{3.7}$$

We shall first construct a sequence of upper bound approximate solutions of (3.1) by means of the following scheme. Let

$$s_0(r,t)=J\left[\left(\frac{r}{m}\right)^l H\left(\frac{m^l N}{I}\right)\right],\ m^l\equiv\max_{[a,b]} r^l, \tag{3.8}$$

where J is the functional inverse of H on $[0,\infty)$, i.e.

$$J[H(z)]=H[J(z)]=z\qquad (z\ge 0). \tag{3.9}$$

Notice that

$$J'(w)=\frac{1}{H'[J(w)]}\qquad (w>0).$$

Subsequent approximations are defined recursively by the equation

$$s_{k+1}+\int_0^t H'(s_k)s_{k+1}\,\mathrm{d}\tau=\frac{r^l}{I}\left(N+\int_0^t\int_a^b H(s_k)q(\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau\right)+\int_0^t [H'(s_k)s_k-H(s_k)]\,\mathrm{d}\tau. \quad (3.10)$$

It is clear that if $s_k \to s$ as $k\to\infty$, then the formal limit of (3.10) reduces to (3.1). To establish this rigorously, we shall first show that

$$\text{(a)}\ s_k>0,\quad \text{(b)}\ s_{k+1}\le s_k\ (k=0,1,2,\ldots;\ t>0). \quad (3.11)$$

on $[a,b]\times[0,\infty)$. Our basic tool is the elementary fact that if v satisfies the equation

$$v(t)+\int_0^t Q(\tau)v(\tau)\,\mathrm{d}\tau=f(t)\qquad (t\ge 0) \quad (3.12)$$

where $f\ge 0$ is continuous and nondecreasing on $[0,\infty)$ and $Q\ge 0$ and continuous, then

$$v(t)\ge f(t)\exp\left[-\int_0^t Q\,\mathrm{d}\tau\right]\qquad (t\ge 0).$$

Thus $v\ge 0$ on $[0,\infty)$. If, in addition, $f>0$ on $(0,\infty)$, then $v>0$ on $(0,\infty)$. If, on the other hand, $f\le 0$ and is nonincreasing on $[0,\infty)$, then $v\le 0$ on $[0,\infty)$.

Since (3.10), thought of as an equation in s_{k+1}, is of the form (3.12), it follows from (3.5), (3.6) and (3.7) that if $s_k>0$ for $t>0$, then so is s_{k+1}. But, the fact that $s_0>0$ for $t>0$ is immediate from (3.8). Therefore (3.11) (a) is established.

We first verify (3.11) (b) for the case $k=0$. In fact, setting $k=0$ in (3.10) and applying (3.8), (3.9) we get

$$s_1+\int_0^t H'(s_0)s_1\,\mathrm{d}\tau=\frac{r^lN}{I}+\int_0^t H(s_0)\,\mathrm{d}\tau+\int_0^t [H'(s_0)s_0-H(s_0)]\,\mathrm{d}\tau. \quad (3.13)$$

Here we have used a fact which will be applied repeatedly in what follows. Namely,

$$\frac{r^l}{I}\int_0^t\int_a^b H(s_0)q(\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau=\int_0^t H(s_0)\,\mathrm{d}\tau.$$

Subtracting s_0 from both sides of (3.13), we obtain

$$s_1-s_0+\int_0^t H'(s_0)(s_1-s_0)\,\mathrm{d}\tau=\frac{r^lN}{I}-s_0. \quad (3.14)$$

It must be shown that the right-hand side of (3.14) is nonpositive and nonincreasing on $[0,\infty)$.

In fact, for $t\ge 0$,

$$\begin{aligned}\frac{r^lN}{I}-s_0&=\frac{r^lN}{I}-J\left[\left(\frac{r}{m}\right)^l H\left(\frac{m^lN}{I}\right)\right],\\ &\le\frac{r^lN}{I}-\left(\frac{r}{m}\right)^l J\left[H\left(\frac{m^lN}{I}\right)\right],\\ &\le\frac{r^lN}{I}-\left(\frac{r}{m}\right)^l\frac{m^lN}{I}\le 0.\end{aligned}$$

The second step above is justified by the inequality

$$J(\lambda w)>\lambda J(w)\qquad (0<\lambda<1,\ w>0), \quad (3.15)$$

which is deduced from (3.7) as follows. Let $\mu \geq 1$, fix $z > 0$ and define

$$\phi(\mu) = H(\mu z) - \mu H(z).$$

Clearly, $\phi(1) = 0$ and for $\mu > 1$,

$$\phi'(\mu) = H'(\mu z)z - H(z) > H'(z)z - H(z) > 0$$

by (3.7). This proves that

$$H(\mu z) > \mu H(z) \qquad (\mu > 1,\ z > 0). \tag{3.16}$$

In order to obtain (3.15) from (3.16), we apply J to both sides of (3.16) to get

$$\mu z > J(\mu H(z)).$$

Now let

$$w = \mu H(z), \quad \lambda = \frac{1}{\mu},$$

and the above inequality becomes (3.15).

For the completion of the proof that $s_1 \leq s_0$, we observe that for $t > 0$, $r' > 0$,

$$\begin{aligned} \frac{r^l\dot{N}}{I} - \dot{s}_0 &= \frac{r^l\dot{N}}{I} - \frac{1}{H'\left[J\left[\left(\frac{r}{m}\right)^l H\left(\frac{m^l N}{I}\right)\right]\right]}\left(\frac{r}{m}\right)^l H'\left(\frac{m^l N}{I}\right)\frac{m^l}{I}\dot{N}, \\ &\leq \frac{r^l\dot{N}}{I} - \frac{1}{H'\left(\frac{m^l N}{I}\right)}\left(\frac{r}{m}\right)^l H'\left(\frac{m^l N}{I}\right)\frac{m^l \dot{N}}{I} \quad \leq 0. \end{aligned} \tag{3.17}$$

Here, we have used the fact that both J and H' are increasing on $[0, \infty)$ and that $N > 0$ on $(0, \infty)$. In the special case $a = 0$,

$$s_0(0, t) = s_1(0, t) = 0.$$

Having confirmed (3.11) (b) for $k = 0$, we now suppose that $s_{k-1} \geq s_k$ for some $k \geq 1$, and use (3.10) to write

$$\begin{aligned} s_k - s_{k+1} + \int_0^t H'(s_k)(s_k - s_{k+1})\,\mathrm{d}\tau &= \frac{r^l}{I}\int_0^t \int_a^b [H(s_{k-1}) - H(s_k)]q(\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau \\ &+ \int_0^t [H'(s_{k-1})(s_{k-1} - s_k) - H(s_{k-1}) + H(s_k)]\,\mathrm{d}\tau. \end{aligned} \tag{3.18}$$

Since the right-hand side of this equation is initially zero, one need only verify that its derivative is nonnegative for $t > 0$. This is obviously the case for the double integral term. As for the single integral, it follows from the Mean Value Theorem that at points where $s_{k-1} > s_k$,

$$H'(s_{k-1})(s_{k-1} - s_k) - [H(s_{k-1}) - H(s_k)] = [H'(s_{k-1}) - H'(\xi)](s_{k-1} - s_k)$$

for some

$$s_k < \xi < s_{k-1}.$$

Since H' is nondecreasing, the desired positiveness follows from the induction hypothesis.

By virtue of (3.11), the sequence s_k converges pointwise to a function s on $[a, b] \times [0, \infty)$. In order to see that s actually satisfies (3.1), we must determine the smoothness of each of the s_k. It is clear from the definition (3.8) of s_0 and the assumed properties of H and N (also r^l is assumed to be C^1 on $[a, b]$) that $s_0(r, t)$ is continuous on $[a, b] \times [0, \infty)$. By an elementary iteration argument using (3.10), one can show that if s_k is continuous in this domain, then the same must be true for s_{k+1}. Thus all members of the sequence are continuous in $[a, b] \times [0, \infty)$.

The first consequence of this fact is that, by the Monotone Convergence Theorem, s is integrable and, in taking the limit of (3.10) as $k \to \infty$, one can exchange limits with integrals. Therefore, s rigorously satisfies (3.1). Furthermore, we may apply Dini's Theorem in exactly the same manner as in [4] to see that $\{s_k\}$ converges uniformly to s in any region $[a, b] \times [0, T]$. Thus, s is continuous in $[a, b] \times [0, \infty)$. Therefore, by (3.1), so is $\dot{s}$.

In order to construct a sequence $\{\sigma_k\}$ of lower bounds, we define

$$\sigma_0(r, t) = J\left[\left(\frac{r}{\mu}\right)^l H\left(\frac{\mu^l N}{I}\right)\right], \quad \mu^l \equiv \min_{[a,b]} r^l; \tag{3.19}$$

$$\sigma_{k+1} + \int_0^t \left[\frac{H(s_0) - H(\sigma_k)}{s_0 - \sigma_k}\right]\sigma_{k+1}\, d\tau = \frac{r^l}{I}\left(N + \int_0^t \int_a^b H(\sigma_k) q(\xi)\, d\xi\, d\tau\right) + \int_0^t \left[\frac{H(s_0)\sigma_k - H(\sigma_k)s_0}{s_0 - \sigma_k}\right] d\tau \tag{3.20}$$

($k = 0, 1, 2, \ldots$), where s_0 is given by (3.8). It is asserted that†

$$\sigma_k < s_0, \qquad \sigma_k \le \sigma_{k+1} \qquad (k = 0, 1, 2, \ldots; t > 0). \tag{3.21}$$

For the verification that

$$\sigma_0 < s_0 \qquad (a \le r \le b,\ t > 0) \tag{3.22}$$

we first let $\mu^l < r^l < m^l$. Then (3.16) implies

$$\sigma_0(r, t) < J\left[H\left(\frac{r^l N}{I}\right)\right] = \frac{r^l N}{I} \quad (t > 0). \tag{3.23}$$

But by (3.8) and (3.15),

$$s_0(r, t) > \left(\frac{r}{m}\right)^l J\left[H\left(\frac{m^l N}{I}\right)\right] = \frac{r^l N}{I} \qquad (t > 0). \tag{3.24}$$

If either $r = \mu$ or $r = m$, at least one of the above inequalities remains strict.

In order to see that $\sigma_0 \le \sigma_1$ we set $k = 0$ in (3.20) to obtain

$$\sigma_1 + \int_0^t \left[\frac{H(s_0) - H(\sigma_0)}{s_0 - \sigma_0}\right]\sigma_1\, d\tau = \frac{r^l N}{I} + \int_0^t \frac{\sigma_0[H(s_0) - H(\sigma_0)]}{s_0 - \sigma_0}\, d\tau.$$

Here, we have used the fact that

$$\frac{r^l}{I}\int_0^t \int_a^b H(\sigma_0) q(\xi)\, d\xi\, d\tau = \int_0^t H(\sigma_0)\, d\tau. \tag{3.25}$$

Therefore,

$$\sigma_1 - \sigma_0 + \int_0^t \left[\frac{H(s_0) - H(\sigma_0)}{s_0 - \sigma_0}\right](\sigma_1 - \sigma_0)\, d\tau = \frac{r^l N}{I} - \sigma_0.$$

†If $a = 0$, we define $\sigma_k(0, t) = 0$ ($k = 1, 2, \ldots; t \ge 0$), and use (3.19), (3.20) for $r > 0$. In what follows, we shall assume for simplicity that $a > 0$.

By (3.16) and (3.19),

$$\frac{r^l N}{I} - \sigma_0 = \frac{r^l N}{I} - J\left[\left(\frac{r}{\mu}\right)^l H\left(\frac{\mu^l N}{I}\right)\right]$$
$$\geq \frac{r^l N}{I} - J\left[H\left(\frac{r^l N}{I}\right)\right] = 0.$$

Also, for $t > 0$,

$$\frac{r^l \dot{N}}{I} - \dot{\sigma}_0 = \frac{r^l \dot{N}}{I} - \frac{1}{H'\left(J\left[\left(\frac{r}{\mu}\right)^l H\left(\frac{\mu^l N}{I}\right)\right]\right)}\left(\frac{r}{\mu}\right)^l H'\left(\frac{\mu^l N}{I}\right)\frac{\mu^l \dot{N}}{I},$$
$$\geq \frac{r^l \dot{N}}{I}\left(1 - \frac{1}{H'\left(J\left[H\left(\frac{\mu^l N}{I}\right)\right]\right)} H'\left(\frac{\mu^l N}{I}\right)\right) = 0.$$

In both cases, strict inequality holds when $r^l > \mu^l$.

We have thus established (3.21) for the case $k = 0$. It must now be confirmed for arbitrary $k \geq 1$, given that

$$\sigma_{k-1} < s_0,\ \sigma_{k-1} \leq \sigma_k \qquad (t > 0). \tag{3.26}$$

Replacing k by $k-1$ in (3.20) and subtracting

$$s_0 + \int_0^t \left[\frac{H(s_0) - H(\sigma_{k-1})}{s_0 - \sigma_{k-1}}\right] s_0 \, d\tau$$

from both sides of the resulting equation, we get

$$\sigma_k - s_0 + \int_0^t \left[\frac{H(s_0) - H(\sigma_{k-1})}{s_0 - \sigma_{k-1}}\right](\sigma_k - s_0)\, d\tau = \frac{r^l N}{I} - s_0 + \frac{r^l}{I}\int_0^t \int_a^b [H(\sigma_{k-1}) - H(s_0)] q(\xi)\, d\xi\, d\tau. \tag{3.27}$$

Here, the equation before (3.14) has been used. It has already been shown in the course of constructing the upper bounds that

$$\frac{r^l N}{I} - s_0 \leq 0,\ \frac{r^l \dot{N}}{I} - \dot{s}_0 \leq 0.$$

Since the integrand of the double integral in (3.27) is strictly negative for $t > 0$ by (3.26), the first inequality in (3.21) is established.

For the second inequality, we rewrite (3.20) in the form

$$\sigma_{k+1} + \int_0^t \left[\frac{H(s_0) - H(\sigma_k)}{s_0 - \sigma_k}\right](\sigma_{k+1} - \sigma_k)\, d\tau = \frac{r^l}{I}\left(N + \int_0^t \int_a^b H(\sigma_k) q(\xi)\, d\xi\, d\tau\right) - \int_0^t H(\sigma_k)\, d\tau.$$

Replacing k by $k-1$ in this equation and taking the difference, we get

$$\sigma_{k+1} - \sigma_k + \int_0^t \left[\frac{H(s_0) - H(\sigma_k)}{s_0 - \sigma_k}\right](\sigma_{k+1} - \sigma_k)\, d\tau = \frac{r^l}{I}\int_0^t \int_a^b [H(\sigma_k) - H(\sigma_{k-1})] q(\xi)\, d\xi\, d\tau$$
$$+ \int_0^t \left[H(\sigma_{k-1}) - H(\sigma_k) + \left[\frac{H(s_0) - H(\sigma_{k-1})}{s_0 - \sigma_{k-1}}\right](\sigma_k - \sigma_{k-1})\right] d\tau.$$

Clearly, it suffices to check the sign of the integrand of the single integral term on the right-hand side above. In fact, suppose that at point (r, τ), $\sigma_k > \sigma_{k-1}$. Then,

$$H(\sigma_{k-1}) - H(\sigma_k) + \left(\frac{H(s_0) - H(\sigma_{k-1})}{s_0 - \sigma_{k-1}}\right)(\sigma_k - \sigma_{k-1})$$

$$= \frac{1}{s_0 - \sigma_{k-1}}[(H(\sigma_{k-1}) - H(\sigma_k))(s_0 - \sigma_{k-1}) + (H(s_0) - H(\sigma_{k-1}))(\sigma_k - \sigma_{k-1})]$$

$$= \frac{1}{s_0 - \sigma_{k-1}}[(H(\sigma_{k-1}) - H(\sigma_k))(s_0 - \sigma_k) + (H(s_0) - H(\sigma_k))(\sigma_k - \sigma_{k-1})]$$

$$= \frac{(s_0 - \sigma_k)(\sigma_k - \sigma_{k-1})}{s_0 - \sigma_{k-1}}[H'(\zeta_0) - H'(\zeta_1)]$$

where

$$\sigma_{k-1} < \zeta_1 < \sigma_k < \zeta_0 < s_0.$$

The integrand is therefore positive at (r, τ) due to the induction hypothesis, the first part of (3.21), and the convexity of H.

It has now been established that $\{\sigma_k\}$ is monotone nondecreasing the bounded from above. Therefore the pointwise limit σ exists. Again, uniform convergence is easily confirmed, so that σ and $\dot{\sigma}$ are both continuous in $[a, b] \times [0, \infty)$. It now appears that we have constructed two solutions, s and σ, of (3.1). In fact, (3.1) has at most one continuous solution. The reader can convince himself of this by inspecting the uniqueness argument in [8] ($m = 0$), which is easily adapted to (3.1) under less restrictive hypotheses than those of this section. Using s to denote the unique solution of (3.1), we have

$$\sigma_k \le s \le s_k \quad (k = 0, 1, 2, \ldots). \tag{3.28}$$

For example, let N be constant for $t \ge 0$ and set $k = 1$ in (3.28). The resulting inequalities are

$$s(r, t) \le s_0(r) + \left[\frac{r^l N}{I} - s_0(r)\right] \exp\left[-tH'(s_0)\right], \tag{3.29}$$

$$s(r, t) \ge \sigma_0(r) + \left[\frac{r^l N}{I} - \sigma_0(r)\right] \exp\left[-t\left(\frac{H(s_0) - H(\sigma_0)}{s_0 - \sigma_0}\right)\right], \tag{3.30}$$

where s_0 is given by (3.8) and σ_0 by (3.19).

4. CONCLUSIONS

A priori upper and lower bounds such as those derived above have various practical applications. In those engineering situations where only rough estimates of stresses or deformations are required, simple bounding formulas furnish quick results which would otherwise require expensive and time-consuming computer work. For example, bounds can be used to provide *a priori* guidelines for a preliminary design which is later subjected to detailed numerical analysis.

Furthermore, bounds and monotonicity results such as (2.20) and (2.21) can be used to check the quality of experimental data and computer output. A numerical solution which significantly exceeds mathematically proven bounds or which fails to exhibit a predicted monotonicity would have to be rejected. Even in cases where a very complicated problem is to be solved using a commercial creep "package", the latter can be evaluated by means of a trial run on a simpler problem for which good theoretical bounds exist.

Finally, the monotone sequences on which the bounds of Section 3 were based can, themselves, be used as the basis for numerical schemes. They provide an alternative to the incremental schemes traditionally used for creep work. They could even, in practice, be applied in situations where the hypotheses of Section 3 do not hold, provided that numerical convergence results or that the formal steady state solutions are achieved after some time.

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